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Developing Understanding of Geometry and Space in the Primary Grades

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Our approach to geometry with young children begins with students' informal knowledge about situations, followed by progressive mathematical reinterpretation of these experiences, an approach consistent with the Dutch approach to "realistic mathematics education" (see Gravemeijer, chap. 2, this volume). Young children's everyday activities—looking, walking, drawing, building, and manipulating objects—are a rich source of intuitions about spatial structure (Freudenthal, 1983; Piaget & Inhelder, 1948/1956; Streefland, 1991; van Hiele, 1986). By looking at pattern and form in the world, children develop informal knowledge about geometric constructs like perspective, symmetry, and similarity. For example, preschoolers pretend that miniatures are small-scale versions of familiar things, and even infants distinguish contour and symmetry (Fantz, 1958; Gravemeijer, chap. 2, this volume; Haith, 1980). By walking in their neighborhoods, children learn to reason about landmarks, routes, and other elements of large-scale space (Piaget, Inhelder, & Szeminska, 1960; Siegel & White, 1975). By drawing what they see, children represent form (Goodnow, 1977). By building structures with blocks, toothpicks, or Tinkertoys, children experience first-hand how shape and form play roles in function (e.g., objects that roll vs. those that do not) and structure (e.g., sturdiness; see Middleton & Corbett, chap. 10, this volume).

Everyday experiences like these, and the informal knowledge children develop over time by participating in them, constitute a springboard into geometry. For example, the ideas that children develop about position and direction while walking in their neighborhood can be elaborated mathematically in a variety of ways—as coordinate systems, as compass directions, as maps, and as dynamic Logo models. Each of these mathematical forms of thought has antecedents in children's experiences (e.g., maps in children's drawings, coordinate systems in city blocks), and collectively these experiences constitute a good grounding for making mathematical sense of the spatial world.

Although rooting mathematics in children's experiences is consistent with theories of learning that emphasize the importance of situation (Lave & Wenger, 1991), it is essential that teachers also establish a classroom culture that grounds student activity in mathematical reflection and generalization (Cobb, Yackel, & Wood, 1995; Vygotsky, 1978; Watt, chap. 17, this volume; Wertsch, 1991). Consequently, developing student understanding of geometry relies on classoom culture as much as it does on mathematically fruitful situations. Skilled teachers develop models of student cognition and its typical trajectories of change (Clark & Peterson, 1986; Fennema & Franke, 1992; Schifter & Fosnot, 1993). Such models help teachers recognize "teachable moments" and other worthy landmarks in the ebb and flow of classroom activity.

Teaching and learning, then, are best viewed in tandem. It is important to identify mathematically important ideas and to build on children's experiences in ways that help children see mathematics as a way of making (more) sense of their experiences. Yet it is equally important that teachers understand landmarks in the progression of children's learning because, without a model of student learning, teachers must rely exclusively on curriculum and its associated scripts. Curriculum, however, cannot be designed to meet the manifold of possibilities inherent in a classroom. Hence, no matter how soundly designed, and no matter how sensitive to children's informal knowledge, curriculum alone cannot result in significant conceptual change.

Because many of the developmental trajectories we observed in the longitudinal investigation described in the previous chapter were comparatively "flat" or incremental, we decided to design classroom environments that would promote development of student reasoning about space and geometry. To establish a robust coordination between teaching and learning about space for young children, we collaborated with a small group of primary-grade teachers to develop a primary-grade geometry based on children's everyday activity related to (a) perception and use of form (e.g., noticing patterns or building with blocks), leading to the mathematics of dimension, classification, transformation; (b) wayfinding (e.g., navigating in the neighborhood), leading to the mathematics of position and direction; (c) drawing (e.g., representing aspects of the world), leading to the mathematics of maps and other systems for visualizing space; and (d) measure (e.g., questions concerning how far? how big?), leading to the mathematics of length, area, and volume measure. Our selection of these strands of experience was guided partly by the developmental trends in children's reasoning evident in the results of the longitudinal investigation described in Lehrer, Jenkins, and Osana (chapter 6, this volume) and partly by our intuitions about fruitful continuities between children's experiences and early geometry. The longitudinal investigation provided starting points for development and some potential signposts of

progress, but for the most part we designed instructional environments incrementally, conjecturing about appropriate instructional activities and then testing them in the crucible of the classroom. Collectively, teachers and researchers conducted "design experiments" (Brown, 1992) that revealed typical patterns and progressions of student thinking when students were immersed for prolonged periods of time in classroom activity that supported the development of spatial reasoning. Each year, we revised our instructional design by updating our selection of curriculum tasks and tools, our models of student thinking, and our assessment practices in light of what we learned. In this chapter, we summarize some of the design principles and outcomes of a 3-year study of teaching and learning geometry in several second-grade classrooms.

INSTRUCTIONAL DESIGN

Our instructional design was multicomponential. The key constituents of the design (see Fig. 7.1) included (a) researcher descriptions of student thinking derived from the longitudinal study of development (Lehrer, Jenkins, & Osana, chapter 6, this volume), (b) teacher–researcher collaborative investigation of student thinking in the context of classroom instruction, (c) professional development workshops, and (d) a parent program that enlarged the learning community beyond the walls of the classroom. We briefly describe the first three components; a fuller description of the parent component is available in Lehrer and Shumow (in press).

Portraits of Student Thinking

We developed text and video descriptions of the growth and change in student thinking that we had observed during the longitudinal study and supplemented these descriptions with other research findings, as needed

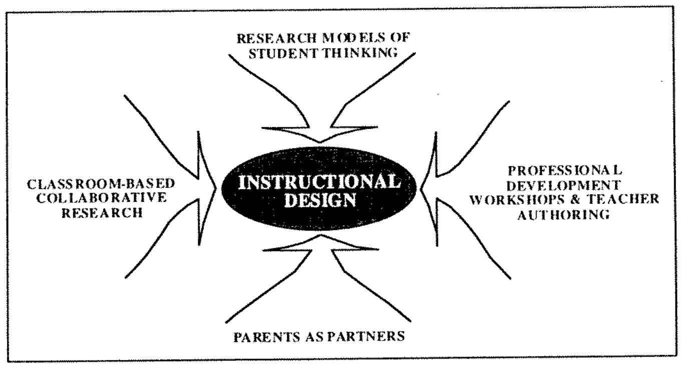

FIG. 7.1. Instructional design for the teaching and learning of geometry.

(see Clements & Battista, 1992, for a summary of related research). The resulting portraits of student thinking traced landmarks in reasoning about (a) structure and form, (b) large-scale space (wayfinding), (c) representations of space including drawings and maps, and (d) measure of length and area (Lehrer, Fennema, Carpenter, & Ansell, 1992). We also collaborated with a small group of teachers to develop descriptions of transitions in student thinking observed in their classrooms. This collaboration resulted in substantial modification and revision of the documents developed by reseachers. What started as a researcher-centered effort was transformed during the course of the 3 years into a collaborative research effort in which teachers and researchers jointly developed portraits of student thinking.

Professional Development

A major portion of the instructional design was devoted to professional development, including in-service workshops, teacher design of curriculum tasks, and teacher-authored descriptions of student thinking related to those tasks.

In-Service Workshops. During the first year, in-service workshops were anchored in Cognitively Guided Instruction (CGI; Carpenter & Fennema, 1992). The CGI program provides teachers with examples of developmental benchmarks in children's acquisition of knowledge about number, especially about situations involving addition and subtraction or multiplication and division. CGI focuses explicitly on the interactive roles of problem structure (e.g., the semantics of types of arithmetic word problems) and solution strategies (e.g., how children solve different types of problems) and, more tacitly, on the role of the teacher as a facilitator of student discourse about mathematical problem solving. In CGI classrooms, teachers commonly base instructional decisions on knowledge of student thinking (Fennema, Franke, Carpenter, & Carey, 1993). Part of the teacher's knowledge is knowledge of the individual child—his or her learning history—and part is knowledge of typical developmental trajectories. Based on developmental benchmarks, teachers decide what is potentially fruitful for "rousing to life" (Tharp & Gallimore, 1988) and what is likely to be pointless and even frustrating to a student. We redesigned CGI to include descriptions of children's spatial reasoning and geometric thinking (Lehrer & Jacobson, 1994).

Ten primary grade teachers participated in approximately 60 hours of in-service workshops devoted to developing their knowledge about key transitions in children's reasoning about space and number. The workshops were led by a CGI mentor teacher, Annie Keith, and a CGI project researcher, Ellen Ansell. Workshop participants read text about the development of children's understanding, saw videos of children solving problems in clinical interview settings, attempted to track the development of thinking in three "target" children in their classrooms, solved problems from the perspective of their children through role playing and

related techniques, and discussed issues related to classroom implementation. Because teachers taught in different buildings, an electronic network facilitated communication among teachers, workshop providers, and researchers.

Tracking the development of selected children during the course of the year proved especially powerful, perhaps because teachers had the opportunity to reason about specific cases rather than reasoning about more general, and perhaps more ambiguous, groups like "children" or "the classroom." Case-based reasoning (Williams, 1992) helped teachers see the relationship between instances of thinking embedded in clinical interviews and children's ongoing thinking in their classrooms. Classroom cases helped teachers instantiate the more general benchmarks about children's thinking presented and discussed during the workshops.

The geometry workshops acquainted teachers with new elements of curriculum (e.g., wayfinding and three-dimensional form), new tools(e.g., magnetic compasses, Polydrons, and Logo), new forms of mathematical notation (e.g., two-dimensional "nets" to represent three-dimensional structures), and what research suggested about developmental benchmarks of student reasoning. Problems that teachers solved were designed to illustrate the potential interplay among instructional tasks, tools, and the notations that we anticipated that students might invent or appropriate. For example, to help teachers recast their understanding of spatial pattern (conceived by most teachers as the result of arranging wooden "pattern blocks" representing familiar forms like squares and trapezoids to form sequences like square, triangle, square, triangle, etc.), during one teacher workshop we presented a task that included the creation of a pattern as a repetition of identical elements (Senechal, 1990):

> Write directions to tell someone how to draw one of the shapes from the pattern-block bucket. What pattern do you notice about the shape, and how is the pattern represented in your description of the drawing? Now write a procedure in Logo to draw the shape. How does the change in representation from paper-and-pencil to Logo change the directions? What stays the same?

While solving this problem, teachers explored simple shapes as a repetition of identical elements and considered the roles that different forms of representation (i.e., drawings, directions, Logo procedures) might play in children's thinking about form and pattern. Logo procedures, for instance, make repetition of identical elements explicit (Lehrer, Randle, & Sancilio, 1989). For example, a square in Logo is the fourfold repetition of a movement of the turtle forward or backward followed by a 90° turn by the turtle. During another workshop, teachers created a map of the school library, a model of a large-scale space. While constructing the map of the library, teachers considered the interplay among spatial configuration, scale, and measurement. Teacher discussion connected this experience to related ideas, such as perspective for depicting objects in the map (bird's-eye view vs. object view) and the utility of symbols for maps (e.g., icons, photos, pictures). Teachers also talked about standard units of measure, what they might be "good for," and what ideas children might have about standard measure.

The in-service workshops helped teachers develop greater understanding of student thinking about number and space, and they introduced teachers to pedagogical issues, especially the importance of multiple forms of representation and the need to help children develop a language about space. The workshops were forums for constructing knowledge, for enlisting the support of peers and of researchers, and for developing a community for the reform of mathematics education in the participating schools. In subsequent years, teachers organized their own forums for professional development, where they gathered to share ideas, reflect on their experiences, and initiate others in their practices. During the third year, this resulted in a teacher-led series of afternoon and weekend workshops for colleagues, with researchers in an advisory role.

Teacher Authoring. After the first year of workshops, five teachers elected to participate in a week-long summer institute, during which they designed tasks to help students develop understanding of the mathematics of space. For each task they designed, teachers conjectured about conceptual landmarks or "ways of thinking" that students would encounter as they worked with the task. Ideally, tasks allowed multiple points of entry for different forms of background knowledge and skills. Tasks also included appropriation of curriculum units developed elsewhere, with revisions guided by the goal of making student thinking more visible. Besides helping teachers track student understanding, making student thinking visible is a necessary first step in creating shared reference for classroom discussions.

Tasks were refined during group conversation. For example, one teacher, Jennie, proposed incorporating length measure into a thematic unit on "How big am I?" The teacher's original idea was to use a story, "How big is a foot?" to illustrate how using a different unit of measure can change the measured length of an object. She proposed that students use their own feet to measure the length and width of their classroom and then talk about why their measured lengths were different. During the group discussion, several colleagues noted that the task as designed might, in accord with the research portraits of student thinking, raise other issues in measure theory, such as the need for identical units, iteration of units (some teachers thought that students might leave spaces between footprints), and the relationship between the object being measured and the unit of measure. This discussion resulted in redesign of the task to include other personal units of measure and more opportunities for developing the need for standard units (to reconcile different but valid measures obtained with personal units). Jennie then used the task during the course of instruction the following year, documenting benchmarks in student reasoning. The same teachers also participated in a second summer institute the following year, editing and revising their work in light of student performance during the course of the second year. During this second year, teachers' descriptions of student "ways of thinking" became richer and more complete, resulting in more articulated portraits of student thinking.

RESEARCH DESIGN

Our research design reflected the complexity of the instructional design. Accordingly, we developed multiple sources of data about student learning, teacher beliefs and teacher practices. We focused on three Grade 2 classrooms. At the beginning of the study, one teacher had taught for 3 years, another for 7, and the third for 5 years. We tracked change for 3 years in two of the classrooms, and for 2 years in the other classroom (the teacher moved at the beginning of the third year). Two of the classrooms were in the same elementary school; the third was in an elementary school located nearby. The students came from a mixture of middle- and working-class families.

Teacher Practices

We observed each teacher's classroom practices three times each week during the course of each school year. Each observation lasted approximately 2 hours and included audiotape and/or videotape of classroom activity, interviews with individuals or groups of children, (often) interviews with teachers about their goals and their views about each lesson, and artifacts of classroom work. Consequently, we were able to amply document teacher practices and transition during the 3-year span. The classroom observations also provided a window to student understanding and to conceptual change.

Student Learning

Each year, we assessed student learning about (a) two- and three-dimensional shapes, including transformations of the plane; (b) position, direction, and perspective; (c) notations and representations of space, including drawings, plan views, maps, and nets; and (d) measure of length, area, and volume. We also assessed student learning about number, especially student ability to solve a variety of word problems. We employed two forms of individual assessment, administered at the beginning and end of each school year. The first consisted of paper-and-pencil measures of problem solving. The second consisted of clinical interviews that allowed us to examine children's reasoning and strategies in greater depth. Sample items and interview questions are displayed in the Appendix. In addition to this individual level of analysis, classroom observations provided a window to collective student learning as revealed by children's whole-group conversations and by ongoing researcher–student interviews of selected children in each class.

CLASSROOM IMPLEMENTATION

Although the three teachers we observed all participated in the same series of workshops and engaged in the summer institutes devoted to curriculum design, each developed a unique stance in relation to teaching children about space, so that the view from each classroom was characterized more by variation than commonality in activity. For one teacher, Ms. C., the predominant theme for coming to know about space was to measure it, so in her classroom, many of the tasks posed to students involved measurement. For example, students designed their own tape measures for length, investigated and invented units for area, and, given a single sheet of 8.5 × 11 in. paper, designed popcorn containers that would hold the most and the least popcorn. In contrast, for Ms. S., the predominant theme for coming to know about space was to experiment with form, and many of the tasks she posed to students involved contrasting and comparing different two- and three-dimensional forms, finding and constructing the Platonic solids, and designing quilts and other patterns. For Ms. J., children's learning about space was anchored in wayfinding and representation. Accordingly, she most often posed tasks to children that involved wayfinding, mapping, and graphing.

Transitions in Task Structure

Despite the difference in the strand of spatial mathematics that teachers chose as their "leading edge" or emphasis, all teachers shared some core practices. Over time, all three teachers went from posing tasks in isolation to developing sequences of tasks that provided children with opportunities for progressive elaboration of core concepts. For example, in Ms. J.'s first year, children wrote directions to find a spot marked "x" in their classroom, with revisions emerging as children negotiated the measuring of length and turns in their directions. By the third year, writing directions was merely the starting point for a series of tasks involving wayfinding, mapmaking, and Logo. Consequently, children in her class were afforded progressive opportunity to reconstitute their initial understanding of position and direction from intrinsic notions of position and direction, to understandings that encompassed perspective, scale, and extrinsic frames of reference.

Transitions in Representational Fluency

During the course of the study, teachers also increasingly emphasized representational fluency; children invented or appropriated multiple forms of representation. For example, in Ms. S.'s classroom, students reconstituted their natural language for two- and three-dimensional forms as properties; they transformed their understanding of form to include (consensually agreed on) lists of these properties, and amplified their knowledge of form through drawings (e.g., top, front, and side views of solids), Polydron constructions (e.g., trying to find "perfect" [Platonic] solids), and Logo procedures. A related development was a proliferation of tools (and associated practices), which children used to sustain investigation of space (ranging from magnetic compasses and Polydrons to student-invented means for measuring length, area, and volume).

Transitions in Classroom Discourse

A noticeable shift in conversational patterns emerged during the 3 years of the study. Initially, teachers were more likely to encourage classroom talk when it concerned numeric solution strategies: They asked children to compare and contrast the solution strategies of their peers and emphasized that a number of different strategies could be used to find an answer. In contrast, their initial scaffolding of conversations about space generally consisted of simple elicitation of multiple ideas from children, a "stirring" of the pot of ideas,with little attempt to compare and contrast ideas or to guide children toward selecting some of their ideas for further exploration.

During the course of the study, the nature of classroom conversations about space changed dramatically. First, there were many more of them as the proportion of time children spent exploring the mathematics of space increased, especially during the second year of the study. Second, teachers became increasingly adept at discerning patterns in children's thinking about space, so that they often helped children talk about similarities and differences among their ideas, in contrast to their previous pattern of simple elicitation without comparison. Third, teachers became much more adept at helping children develop a coherent language about space to supplement their initial, near-exclusive reliance on gesture and shared visual regard to communicate about space. Children's talk about figures came to include descriptions of properties and ways to generate instances, in contrast to their earlier reliance on single-word descriptions (e.g., "skinny") and gesture. Fourth, talk became intimately connected to justification and argument. Classroom conversations often revolved around the need of students to convince others of the validity of their own viewpoints.

In summary, during the years of the study, teachers either designed or appropriated a number of different tasks as initiators of student learning. Tasks became progressively more interconnected, and teachers used them to revisit or "spiral" important ideas. Representational fluency was increasingly emphasized, so that students rarely ever talked without drawing or building, or measured without designing a tool. Perhaps the most noticeable change was in the nature of classroom talk about space. Children's initial talk about space was nearly always gestural and rarely intersubjective; in later conversations, gesture supplemented linguistic description, and language was often taken as shared.

STUDENT LEARNING

Each year of the study, we measured students' problem solving in both space and number, and we noted significant transitions in student thinking each year in all classrooms. We found significant growth in children's number sense and in children's spatial sense, as indicated by the number of problems correctly answered at the beginning and end of each year.

For number sense, individual interviews suggested that this change in performance could be attributed to two main factors. First, children developed more sophisticated strategies for solving number problems during the year. To measure this increase in sophistication, strategies were assigned levels: 0 for No Solution, 1 for Direct Modeling (e.g., children uses counters to represent quantities following the action sequence in the word problem), 2 for Counting (e.g., children solve an addition problem by counting on from the larger addend), 3 for Recalled Fact, Derived Fact, and Algorithm (e.g., children invent or use algorithms, like $27 + 27 \rightarrow 20 + 20 = 40$ and $7 + 7 = 14 \rightarrow 40 + 14 = 54$; see Fennema et al., 1996). The growth in the highest level strategy available to children at the beginning and end of each year is displayed in Fig. 7.2a. Second, over time, children were able to apply a wider range of strategies to problems. To track progress in the range of strategies available to children, strategies were assigned to one of four classes: Direct Modeling, Counting, Recalled and Derived Facts, and Algorithms. The significant increase in number of classes of strategies that children demonstrated to solve arithmetic word problems is illustrated in Fig. 7.2b.

We also noted significant pre–post conceptual change in each of the four strands of spatial sense, as indicated by student scores on problems

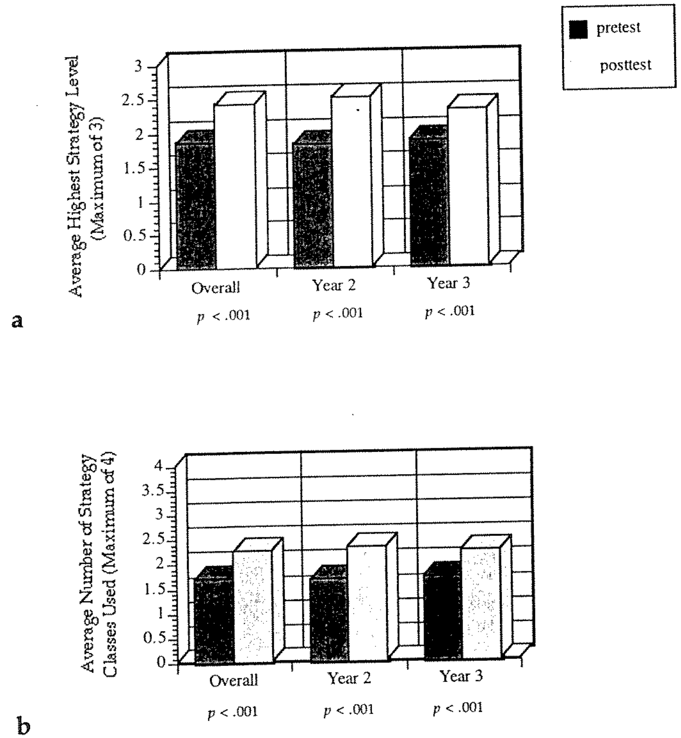

FIG. 7.2. Change in students' use of arithmetic problem-solving strategies over the 3-year study, by year.

administered at the beginning and end of each school year. For example, Fig. 7.3 shows the improvement in students' performance on problem-solving assessments during the third year of the study. The proportion correct was calculated for each of the four strands, and these four proportions were then combined with equal weight to arrive at a total score. Individual interviews suggested major transformations in how children thought about the structure of two- and three-dimensional space, measurement, representations of space, and position and direction in a large-scale space. For example, children's initial conceptions of shape and form were dominated by resemblance to familiar objects and other aspects of appearance (see Lehrer et al., chap. 6, and Pegg & Davey, chap. 5, this volume), but they changed during the course of the year to include reasoning about properties across a variety of contexts. Similarly, children's initial ideas about length or area measure often confused the two, but every year, their understanding of key ideas in length and area measurement, such as the need for identical units of measure, far exceeded that of fifth-grade children in their school. Although these transitions were expected, they were unusual considering the relatively static patterns of growth and development that we observed in the longitudinal study (Lehrer et al., chap. 6, this volume).

TEACHING AND LEARNING IN THE CLASSROOM

To illustrate the interactions between teaching and learning in these classrooms that could account for the striking pattern of conceptual change noted previously, we focus on two strands of learning in Ms. C.'s classroom. In the first, children learned about transformational geometry and symmetry as they designed a quilt. Not all of children's learning was domain specific; children also explored issues in epistemology, especially the the limits of case-based induction. In the second, children learned about area and its measure. The lessons on area illustrate how teachers wove

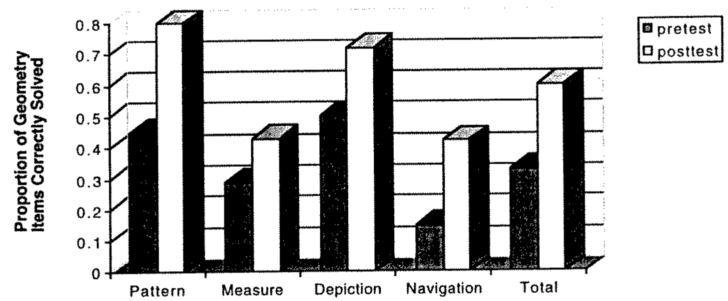

FIG. 7.3. Improvement in students' performance on problem-solving assessments during the 3-year study, by strand of geometric reasoning.

tasks, notations, and conversations to guide children's exploration of the foundations of measure theory.

Designing Quilts[1]

Quilts and quilting were part of the cultural heritage of most of the children in this classroom; children were interested in the quilt designs they saw in books, video, still photos, and pieces brought to the classroom by quilters in the community. Quilt design provided children the opportunity to explore important mathematical ideas like symmetry and transformation, and to develop conjectures about how these mathematical ideas might contribute to their aesthetic experiences of the artistry of quilt designs. Here we trace the progression of student thinking in Ms. C.'s class over a 5-week period during the third year of the study, by focusing on "snapshots" of classroom activity and conversation. More extensive discussion of connections between children's thinking about quilt design and algebraic concepts are discussed in Strom and Lehrer (in preparation).

Partitioning Space. Students first designed a paper-and-crayon (replaced later by paper cutouts and then computer-screen objects) "core square," the basic object subjected to geometric transformations to produce a quilt (see Fig. 7.4). A core square was composed of an array of squares, each partitioned into two right isosceles triangles. For some students, the apparently transparent idea of partitioning the square into triangles was somewhat problematic, partly because the task presupposes acquisition of diagonality (Olson, 1970).

After creating the core square, children used isometries of the plane (flips, turns, and slides) to arrange four identical copies of their core square into 2 × 2 designs. These designs were then composed to create the final pattern for the quilt. Other quilts were designed by transforming "strips" of core squares: A strip was a row of core squares arranged by application of the three transformations.

The complexity of the core square could be varied by using a mosaic of different forms (e.g., smaller triangles that tiled the region) and colors. Efforts to redesign quilts by changing the core square led children to consider a variety of ways of partitioning the same region of space. Moreover, these design efforts helped children explore the consequences of different transformations and combinations of transformations for properties of form like color adjacency and symmetry. In the sections that follow, we describe these and related forms of thinking about the plane.

Distinguishing Physical from Mathematical Motion. Because children constructed two-sided paper core squares or core squares composed of Polydrons, they could physically enact flips, slides, and turns. The curriculum confined these initial experiences to translation, vertical and hor-

[1] The quilt design curriculum was developed by Education Development Center. The classroom teacher described here also contributed to the design of the curriculum during field testing.

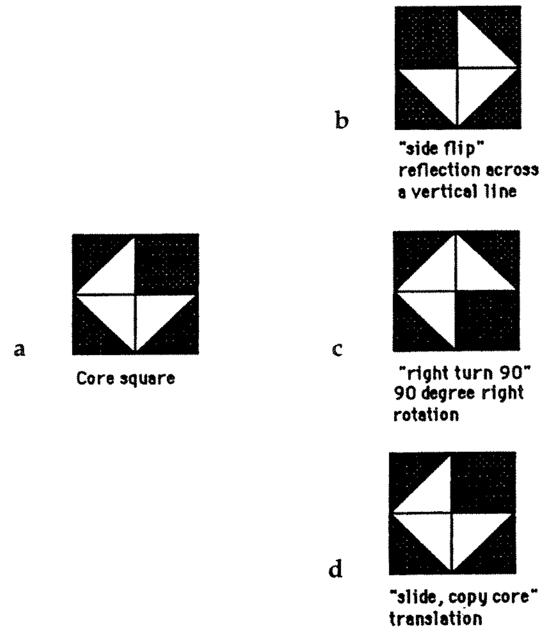

FIG. 7.4 Sample core square and three possible transformations. (a) Core square. (b) "Side flip": reflection across a vertical line. (c) "Right turn 90": 90° right rotation. (d) "Slide, copy core": translation.

izontal reflection (flips), and rotation in increments of 90°. To help children mathematize these physical motions, children developed a notation to describe each motion so that they could easily write directions for other children to follow when replicating a particular quilt design. Through discussion and consensus building, children developed the following notations for flips: UF for *up-flip* (reflecting about the horizontal axis by grabbing the bottom edge and flipping the square over), DF *down-flip* (reflecting about the horizontal axis by grabbing the top edge and flipping the square over), LF *left-flip* (reflecting about the vertical axis by grabbing the right edge and flipping the square over), and RF *right-flip* (reflecting about the vertical axis by grabbing the left edge and flipping the square over). Note that children's notations for reflections included some that could be distinguished in physical motion but that had no mathematical counterparts.

When conjecturing about what steps a fellow student might have taken to create her 2 × 2 design from the original core square, some students thought that one of the actions taken was an up-flip, and other students thought that the same action was a down-flip. Upon testing these conjectures, the class discovered that both flips produced the same result—it was impossible to distinguish one from the other.

Because no consensus was reached after the first example, Ms. C. went on to test other examples, using Polydron models of children's core squares:

> Maybe that was just their core square. Maybe they just had a weird core square. Let's try it with this core square. OK, do they look the same, the way I have them now?

Note that Ms. C.'s comment about "maybe that was just their core square" invited children to consider multiple cases, and, by implication, a search for a negative case, not just single confirming instances in their justification. At this point, Ms. C. had established a routine for testing the conjecture on a given core square. She placed two copies of the same core square side by side, then flipped one up and the other down. When she finished these flips, the core squares were no longer side by side; the one that was flipped up was above the original position; the one that was flipped down, below the original position. It was visually apparent that the actions performed on the two squares were physically different; it was also obvious that both actions resulted in exactly the same pattern on both squares. After several more examples, children became convinced that they needed to change their notation to UDF for *up–down* flip and SF for a *sideways* flip because different physical actions (up or down, right or left) led to identical results. This process helped children distinguish the plane of action from the plane of mathematics.

Viewing the World Through Notation. The use of notation for communicating design also helped children reexamine physical motion from a mathematical point of view. When children wrote directions for 2 × 2 designs that involved compositions of transformations (successive transformations, like a turn followed by a flip), they often only represented one of the motions in notation. For instance, a child wrote SF for a composed motion, like SF TR 1/4. After prolonged discussion and extended exploration, some members of the class proposed that the source of the difficulty was that their wrists could "flip and turn" at the same time—the differentiation implicit in the mathematical notation was not well differentiated in the motion of the wrist. Children used the notation to recast continuous physical motion as discrete steps. More generally, this discovery is a microcosm of the relationship between a model (here, the motions on the plane) and the world (here, the physical movements of hands): Models fit (to a degree) the world, but the world also changes as students view it through the model. Hence, model-fitting is a dual relation.

Generalization. Ms. C. often invited students to make conjectures about transformations by considering the generality of a case generated by one or more students. The signature phrase "Do you think this is true all the time?" was usually followed by a search for other confirming cases and for counterexamples. The grounds of evidence usually consisted of a larger number of cases that clearly fit what the class called a "rule" (a generalization), accompanied by lack of (a failure to find) counterexamples. This type of conjecture–evidence cycle can be illustrated by a conjecture about the number of flips of a core square that would return it to its original position (the order of the up–down flip). The original position was distinguished by a small "x" in the upper left corner, a convention introduced by the teacher to facilitate discussion. When one child suggested that it would take two up–down flips, Ms. C. replied, "Two? Let's try it. Watch. Memorize Katie's core square. This is what it looks like. One [flipping the core square]. Two [flipping it again]."

At this point, children established that the order of the up–down flip was two. Students suggested that 0 and 4 flips would also work, and they tested this conjecture with the core square. Br then suggested that any *even* number would have the same result. Children went on to explore this conjecture, testing a number of cases before one student, Ke, suggested that counting by 2s "as high as you wanted" would have the same result.

The Limits of Case-Based Generalization. Although the class norm for evidence about a conjecture consisted primarily of inductive generalization from positive and negative cases, the limitations of cases were discovered by this class. Ms. C. noted that all children had designed at least one asymmetric core square, and all of these asymmetric core squares were used to create at least one symmetric 2 × 2 design. Despite Ms. C.'s appeal to students' experiences and their positive cases, some students in the class remained unconvinced about the generality of the conjecture. Ms. C. decided to probe children's thinking about the number of cases that might serve:

Ms. C.:	So if we shared 20 or more ways together today that you could start with an asymmetrical core square, and still every time make a symmetrical 2 × 2 design, how many more do you think we'd have to test before we could say you could always do it?
Ni:	Hundreds of hundreds of thousands.
Na:	We would have to test all the core squares in the world.
Ms. C.:	Could we do that?
Class:	No, no.
Na:	We'd have to test all the core squares in the world that are asymmetric.

Children decided that they would have to test every case (an exhaustive procedure), and ultimately decided that this would be neither practically feasible nor even, in principle, possible because, as one child noted, "People are probably making some right now." We believe that reasoning about the limits of induction, a theme that resurfaced later in this classroom, sets the stage for (informal) proof as a form of argument. We are currently working with teachers on forms of instruction that build on this foundation.

Art and Geometry. During the course of 5 weeks, children's appreciation of the aesthetics of quilt design changed markedly. One measure of this transformation was their talk about what they found interesting when they saw a video accompanying the unit, which displayed a variety of quilt designs. Initially, children's talk about design was weighted heavily toward "cool" colors. Over time, their comments began to shift, so that by the end of the unit their talk about "cool" quilts included consideration of the roles played by lines of symmetry, complexity of form (e.g., number of partitions of the core square), and transformations that produced different types of color adjacency. Students also began to notice the constraints inherent in certain design considerations. One child (Danny), for example, cautioned a peer about the constraints certain symmetrical core square designs put on the variation of 2 × 2 designs that can be constructed from them:

> Don't make one color diamond in the middle and all the corners one other color because no matter if you flip it or do anything with it [meaning transformations to the core square to make the 2 × 2 design] it won't work [to produce multiple 2 × 2 designs]... because it is symmetrical all the way.

Fig. 7.5 displays one instance of Danny's general principle. Note that any transformation will produce, in Danny's words, "the only 2 × 2 you can make" from the core.

Summary

Quilt design was a fertile ground for developing and exploring a mathematical model of the plane. Informal knowledge of drawing and aesthetics served as a springboard to mathematical notation and argument, which were mutually constituted in the ongoing activity of the classroom. Children's explorations of transformations and symmetry resulted in reorganization of their informal knowledge of aesthetics and design, thus

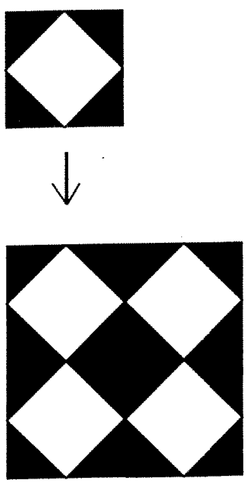

FIG. 7.5. An example of constraint caused by symmetrical design of core square (Danny's general principle). Note that no transformation in design occurs no matter which way the core square is "flipped."

completing a feedback loop in which the mathematization of informal knowledge was incorporated into the body of that knowledge. In the next section, we describe a similar process of progressive mathematization of everyday experience: how children's informal knowledge of appearance and the amount of space covered by a shape was successively transformed into the mathematics of measure.

DEVELOPING AN UNDERSTANDING OF AREA MEASURE

Ms. C. designed a spiral of tasks, all related to thematic units, to help children develop their ideas about area and its measure. The design of each task was guided by her knowledge of how children think about principles of area measure like space-filling, additivity of areas, and identity of units (Lehrer et al., chap. 6, this volume). The progression of tasks illustrates how teachers used their knowledge of children's thinking as a guide for designing and adapting instruction. Children's responses to each task indicate again the interactive roles of tasks, notations, and classroom conversation.

Three Rectangles

Ms. C. first asked children to judge which of three "quilt pieces" (rectangular strips of construction paper tacked on the blackboard) "covers the most space." The dimensions of each quilt piece were unknown to children, but the pieces were designed to correspond to different arrangements of the core squares of the quilting unit (1 × 12, 2 × 6, and 4 × 3 core squares, respectively). The core squares were not demarked in any way. She labeled each quilt with a letter—A, B, and C (see Fig. 7.6).

The design of the task reflected Ms. C.'s understanding of student thinking. She wanted children to construct a unit of area measure by building on their informal knowledge of cutting and rearranging pieces (see Lehrer et al., chap. 6, this volume). She chose these shapes expecting that the conflict between perception (the rectangles appear to cover different amounts of space) and conception would lead to eventual construction of a unit of measure:

> Once they make predictions [about which covers the most space], I expect they might say, "Well, I think Shape C covers more space." "No, no, no, it's A—Look how long it is." But when I ask them, "How can we find out?," what are they going to say? Will they suggest covering it? Will they suggest measuring around the outside? Will they suggest folding it in half? [And I will tell them,] "You are looking at these three shapes. You have different ideas about which might cover more space, but how are you going to prove to someone what you think might be true?" By the end of the lesson, some of them will say, "Well, it looks like it takes up more space, but really you could just push that space around and make it fit."

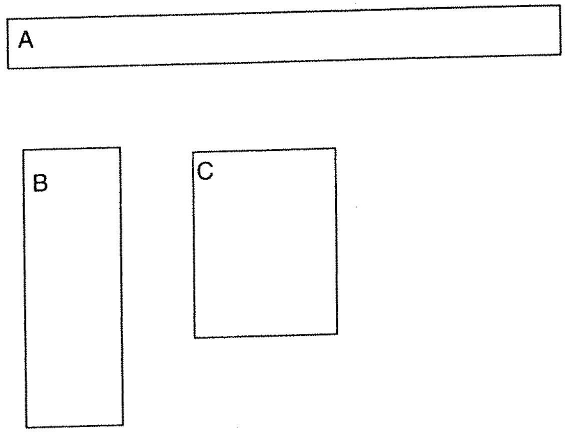

FIG. 7.6. Constructing units of area measure: "Which quilt piece covers the most space?"

Additive Congruence. Most students began by claiming that shape B or shape C would cover the most space because they were "fatter" or "looked bigger." However, some of the students in the class disagreed and thought that perhaps shapes B and C were really the same size. Mi's work is shown in Fig. 7. 7. She decided to fold B in half because she saw that if she folded it in half and rotated it, it would cover exactly half of C. She also noted that if half-B was flipped horizontally, it would cover C. Mi's knowledge of transformations facilitated this exploration. Children went on to explore other partitions of B and C that could lead to this result.

Constructing Units of Measure. Eventually satisfied that B and C did indeed cover the same amount of space, the class turned its attention to A.

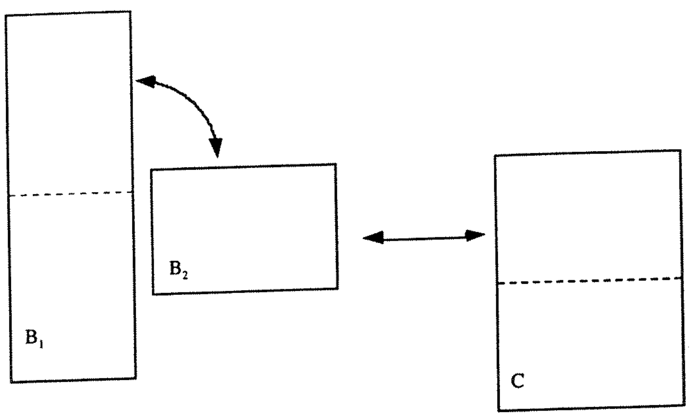

FIG. 7.7. Mi's solution: folding B to measure C.

Mic claimed, "You make C into A," and proceeded to demonstrate by folding C into four equal parts, each the width of A (see Fig. 7.8a). He placed this "strip" at the top edge of A and used his finger to demark the bottom edge. Then he moved the entire strip down to his finger, and proceeded in this way to iteratively mark off four equal segments of A.

Another student, Ti, folded C into three long strips instead of four short strips (see Fig. 7.8b). When Ti finished with his demonstration of additive congruence, and C was unfolded, the fold lines clearly divided the rectangle into a 4×3 array of squares (see Fig. 7.8d). This was not noticed by the class until Ms. C. asked: "How come it took Mic four strips and only took Ti three?"

While the class pondered this problem, Ti was counting "1, 2, 3, . . . , 12." When Ms. C. asked him to clarify what he meant, Ti replied, "Twelve squares! That makes a quilt!"

A second student pointed excitedly to A and said, "Then it takes 12 squares to make *that*." Ms. C.'s question instigated a transition in strategy from additive congruence to units of measure. Before, the children were talking in terms of cutting up shapes and rearranging figures, but at this point they were talking about the number of core squares in a quilt. The children went on to verify that each of the three forms could be composed of exactly 12 square units or core squares. Ms. C. then posed an additional problem of designing as "many shapes as you can" composed of 12 core squares. She used computer software as a tool for letting children freely explore the idea that appearances can be deceiving (e.g., different looking forms can cover the same amount of space) and recognized units of measure as good conceptual tools for addressing this problem.

Ms. C. also used this task as a forum for considering other forms of argument. For example, in another class, one student, Sa, proposed a form of transitive inference—by folding and covering, the class had established that rectangle A and rectangle B each covered the same amount of space (A = B), and also that rectangles B and C each covered the same amount of

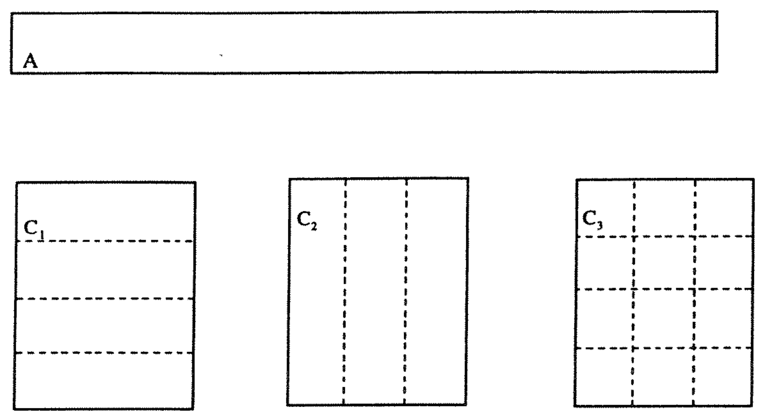

FIG. 7.8. Folding C to measure A.

space (B = C). Consequently, Sa suggested that this must mean that A and C also covered the same amount of space. Ms. C. then asked the class to consider this "without testing," meaning that she wanted them to establish how to verify a conjecture based on a chain of propositions, rather than simply to test its truth empirically.

Ms. C. noted that, at the end of the lesson, children "were using a square in the conventional way (of measuring area) but they were using it in a meaningful way and then they were able to make sense of other area problems ... it's a kind of bridge between a standard unit of measure and a personal unit of measure." Here Ms. C. was drawing a contrast between textbook problems that assume that conventional square units of area measure are conceptually transparent versus her recall of the constructed nature of the square unit in her classroom.

Area of Hands

Following the construction of a unit of measure for the area of a familiar form like a rectangle, Ms. C. posed the problem of rank-ordering the "amount of space covered" by individual students' hands to help children "cement" the utility of a unit of area and to explore the qualities of different potential units of measure. Ms. C. suggested that "the hand doesn't lend itself to thinking about squares of space, as quilts do." (She knew from our research that children prefer units of area measure that perceptually resemble the figure being measured.) She also indicated that the task provided opportunities for children to develop strategies for finding area when the form does not lend itself to easy partitioning into subregions, especially the problem of "what to do with the leftovers" (units that are fractional pieces). The task was ill-structured, in that size, material (real hands or representations), and method were all unspecified.

As Ms. C. anticipated, the task provoked considerable discussion about appropriate units of measure. Children first tried to solve the problem by adapting the strategy that had worked to determine the congruence of the three rectangles: They superimposed pairs of classmates' hands. They discarded this approach as both too time-consuming and perhaps fatally flawed. As one child said, "What do we do with fat thumbs and thin fingers?," indicating that handprints are not uniform, and direct comparisons are, therefore, difficult. This prompted children to consider working from a representation of a hand, rather than working directly with hands. A prolonged discussion eventually led to adoption of a classroom convention about tracing hands on construction paper. Thereafter, children worked with paper representations in lieu of hands.

Children then attempted to develop a unit of measure. As Ms. C. anticipated through her knowledge of children's thinking, most of the children's inventions resembled perceptual features of the hand in some way: beans, fingernails, spaghetti, and rope were all considered and subsequently rejected. Children's reasons for rejection helped them better understand fundamental properties of area measure. For example, children found that using beans wasn't satisfactory: Two different attempts to use beans as units to measure Gl's hand led to two widely different quantities; beans "leave cracks" (a reference to the space-filling principle of area measure); "beans are not all the same" (a reference to the identical-units principle of area measurement); and so on. As children deliberated about their choices, it became clear that they found none of their inventions satisfactory—all violated one or more of the canons of measurement that they had decided were important.

At this point, Ms. C. held up a piece of grid paper marked off in squares and asked children: "Could we use this as a tool?" Most replied very emphatically no, suggesting that squares did not look anything like hands. One child disagreed with the others and said yes, because the squares were all the same (identical units), and they didn't have any "cracks" (space-filling). At this point, another child agreed but noted, "I see a problem—there will be leftovers." After explaining to her peers the nature of the problem, another child proposed a solution: Use different colors to estimate parts that would constitute a whole. For example, $^7/_8$ of one "leftover" might be combined with approximately $^1/_8$ of another to constitute one whole. Both pieces were marked with the same color (e.g., purple), and then other pieces would be identified and marked with different colors (e.g., $^1/_2$ and $^1/_2$ marked red). This system of notation did not quantify the part–whole relation (children never wrote $^1/_8$ or $^1/_2$), but it did help children keep track of their estimates.

By the end of this lesson, children had confronted some fundamental issues involved in constructing a unit of area measure, most especially the need for combining identical units, the importance of space-filling, and the irrelevance of resemblance for judging the merits of a unit of measure. Children invented a system of notation to record their estimates about combining the "leftovers," a process that helped many see that area measure need not be confined to integer values. Most especially, children were able to see measurement do some real work; their deliberations led to a rank-ordering of all of the handprints in the class.

Ms. C. noted that the task also led children to reflect again about area measure, "distinguishing between that kind of measurement [some children first proposed using length measure, e.g., the span of the hand] and what area really is, and they quickly saw that they had to have a way to quantify how much space that [the handprint] covers." As she anticipated, the lesson helped children reflect about the need and functions of units of area measure: "I find that kids, when you do this with them, want something that is going to fill in the tips of the fingers, like beans or fingernails. They wrestle with beans and fingernails, then they figure out that they need things the same size and with complete cover." Ms. C. pointed out that the latter properties of units of area measure (that they be identical and space-filling) were tacit in the core squares of the three rectangles task, but were made explicit in this task.

Area of Islands

Several weeks after rank-ordering the area of hands, children each drew their own "islands," during a unit on geography. They then attempted to rank-order the area of each island. Ms. C. chose this task as a follow-up to the area of the hand, partly because she believed that children needed to explore further the fundamental properties of area measure that they had discovered in the previous lesson ("another context for making sense of why area measure might matter"), and partly because she wanted to introduce children to a notational system for keeping track of the pieces invented in another classroom ("They needed a chance to work with those parts of squares again"). She also believed that the strategy of simply superimposing objects would be more obviously unwieldy, and children would, therefore, be more likely to consider units-of-measure strategies rather than congruence strategies.

Although this lesson was also rich in mathematical talk and provided children further opportunities for exploring principles of area measure, the lesson was perhaps most noteworthy for children's use of their previously invented notations to measure the area of each island. Children's first ideas about measuring area mobilized (Latour, 1990) their previous "color matching" strategy to deal again with the problem of the leftovers (the fractional pieces of area measure), and they reached consensus quickly about the virtues of again employing square-grid paper as a measurement tool. At this point, Ms. C. introduced a new notational system, invented in another class, where students symbolized all part–whole relations (e.g., $^1/_4$, $^1/_2$, $^1/_3$), filled all equivalent fractions with the same color, and then combined the pieces to make whole units. This alternative notational system put another cast on composed congruence: A whole unit could be constituted in a variety of ways, but each of these ways could be shared symbolically, not just indexically. In the first system, there were also multiple ways of making a whole unit, but each composition had to be considered by shared visual regard and represented the judgments of individuals. In the second notational system, each composition was more easily communicated because it could be shared symbolically, and it appealed to conventional representations (i.e., fractional pieces) rather than idiosyncratic representations of part–whole relations.

Ms. C. continued to emphasize understanding and reflection, not simply doing: "[The task provided opportunities for students]" to deepen their understanding of why it mattered for them to think about those leftover parts. And what it was that they were actually doing, which was making those parts into whole units so that you could account for all the space."

Area of Zoo Cages

Children designed a zoo and investigated ways to redesign the city zoo (a project being undertaken by the city). Within this context, Ms. C. posed a problem of comparing the areas of different zoo enclosures on a large sheet of paper displayed on the blackboard (see Fig. 7.9). She suggested that the task provided opportunities for students to revisit the conception–perception mismatch of the three-rectangles task and to re-represent the idea of area measure symbolically, as a multiplication of lengths. In this instance, students were not provided any tools except a ruler.

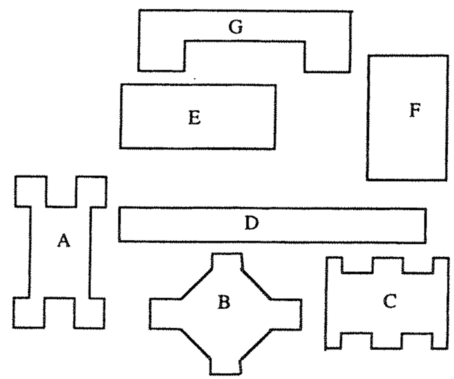

FIG. 7.9. Redesigning a zoo: units of area measure.

The conversation began with one student's assertion that two of the rectangles were "exactly the same." Other students took this as an assertion that the two rectangles were congruent. To test this idea, students proposed superimposing one rectangle on the other, but Ms. C. did not permit students to cut out the rectangles. Another student proposed that if two rectangles were congruent, then their corresponding sides would have the same measure. This student used the ruler and found that the measure of one rectangle was 5 × 8 in. and the other, 4 × 10 in.:

Ms. C.: So the whole Shape E and the whole Shape F—you're saying that Shape E and Shape F are the same?

Ca: Not shapewise, but they take up... they both take up the same amount of space.

Ms. C.: OK. Ca is saying that he thinks Shape E and Shape F cover the same amount of space. He's saying they're not the same shape exactly—he says they look different, and we just measured two sides and showed that they're different.

Ca: Four times 10 is 40 so that means it covers up 40 inches, and then 8 times 5 is 40, so it covers up 40 inches.

Classmates asked Ca what he meant by "covers 40 inches" because what they saw was a pair of rectangles whose longest side was 10 inches. Another child got up and worked with Ca, and together they partitioned each rectangle into 40 square inches, demonstrating two forms of array multiplication: 4 groups of 10 and 10 groups of 4. The conversation then turned to how multiplication of length resulted in units of area, and the class discussed whether or not this principle was true for all rectangles. By the end of the lesson, children had again recast their knowledge of units of area measure—what was formerly known primarily by finding appropriate material means for covering a space had been reconstituted symbolically as multiplicative length.

Student Learning

At the end of each year, we administered items like those posed in the Lehrer et al. (chap. 6, this volume) clinical interviews to children in the three classrooms. The first wave of the Lehrer et al. longitudinal sample provided a baseline for describing second-grade children's conceptions of area measure, and the last wave provided a look at typical patterns of development for these second graders (then in the fourth grade). Against this baseline, we contrasted children's development within the target classrooms as indicated by their performance at the beginning and end of the second grade with respect to their strategies for finding the area of irregular forms, as well as important ideas in area measure, like space-filling (area units should tile the plane). Measurement of these components of area measure at the beginning of the year indicated marked similarity in profiles for the longitudinal and target classroom samples. However, inspection of Fig. 7.10 suggests significant differences by the end of the year; the average performance of children in the target classrooms exceeded that of both waves of the longitudinal sample.

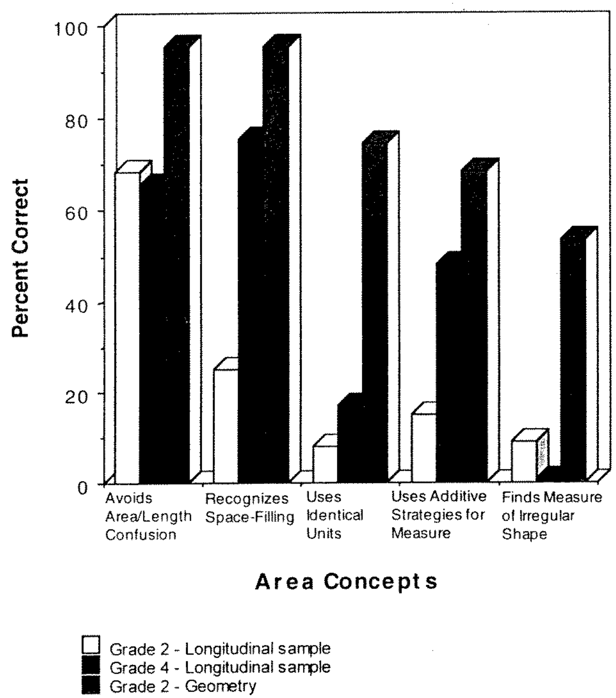

FIG. 7.10. A comparison of children's understandings of area concepts.

Reprise

The sequence of tasks invented by Ms. C. helped children progressively elaborate and mathematize their informal knowledge about area and its measure. The first task, involving comparisons among three rectangles, invited a conflict between perception ("it looks like") and conception (the mathematics of congruence). Ms. C. exploited this tension to motivate practical inquiry built around children's ideas about additive congruence (reallotment, i. e., different arrangements of the same spatial regions do not change area). This practical inquiry took the form of folding and rearranging subregions of the three rectangles, with the eventual emergence of the idea of a unit of measure as children folded one of the rectangles into three congruent pieces vertically and four congruent pieces horizontally. Interestingly, Ms. C. made the equivalence of these two different forms of folding problematic for children; measure emerged from children's resolution of the impasse.

The next two tasks designed by Ms. C. point to the important roles played by tools and notations in the development of understanding of space. When children attempted to rank-order the area of their hands, the very availability of a resource like beans seduced children into investigating its measure properties. Children's discovery of difficulties with beans led to greater understanding of two fundamental principles of area measure: identical units and space-filling (tiling the plane). During their investigation, children invented a notational system that helped them both estimate and record fractional pieces. This notational system was applied again during their investigations of the area of islands, but this time an alternate notation was introduced (symbolic representation of fractional pieces) by Ms. C. Children adopted the alternative notation because it proved easier to conventionalize (the rules about what counted as one unit were less idiosyncratic and more communal) and, therefore, easier to communicate with others. Consequently, children had the opportunity to reflect on the uses and purposes of mathematical notation even as they elaborated their ideas about area and its measure. The last task in the sequence (zoo cages) provided children further opportunity to confront once again the conflict between perception and conception, but this time the resolution was developed at a symbolic, and hence generalizable, plane. Rather than reallot units of measure, children could simply multiply lengths to obtain a single quantity that revealed unambiguously whether or not two shapes "covered the same amount of space."

Collectively, these tasks illustrate a spiral of design that started with children's informal understanding and built successively on the history of the understandings they developed as they solved the problems posed by these tasks. The tasks provided frequent opportunities for emergent goals (e.g., in the three-rectangles task [the first task], issues about squares and rectangles emerged; in the last task, issues about the associativity property of multiplication emerged) even as they provided sufficient structure and constraint for the development of productive mathematical thinking. The tasks also afforded forums (e.g., appearance vs. reality) for the invention of systems of notation in the service of progressive mathematization of everyday experiences. All of the tasks provided frequent opportunties for classroom conversations centered around sharing strategies, making and justifying conjectures, and exploring different forms of mathematical argument.

CONCLUDING COMMENTS

We embarked on a program to redesign geometry education in the primary grades in such a way that young children had the opportunity to develop a mathematics of space even as they were developing a mathematics of number. The cornerstones of our work were commitments to mathematizing children's informal knowledge about space and supporting teachers' professional development. Each teacher devoted substantial resources to develop a professional identity congruent with instruction rooted in understanding children's thinking, and every teacher participated in a small, but generative, community that provided opportunities to elaborate those professional identities. The development of a community of practice was fostered by a series of in-service workshops that emphasized cases of student reasoning, by the conduct of collaborative research, and by teacher authoring.

Teachers' practices reflected a consistent set of design principles. First, teachers invented or appropriated problems and tasks that were rooted in children's informal understandings of space, a practice consistent with the Dutch realistic mathematics education (see Gravemeijer, chap. 2, this volume). For example, children all had ideas about what makes designs like quilts interesting, and their interest in form and pattern provided a rich springboard for the mathematics of transformation and symmetry. Similarly, all children at one time or another experienced conflict between appearance and reality, and teachers skillfully helped children develop the mathematics of area as an explanation for why some forms looked different, yet covered the same amount of space.

Second, teachers continually promoted children's inventions of ways to depict and represent space. These depictions were not merely displays; rather, they were tools for developing mathematical arguments. For instance, children's invented notations for horizontal flips ("up–down" flips) helped them distinguish and describe which aspects of a physical motion were worth preserving mathematically. Other notations connected spatial and number sense: Recall, for instance, children's discussions of odd and even numbers of flips, and their explorations of array multiplication as they thought about measuring areas of rectangles through multiplication of length and width.

Third, teachers promoted forms of classroom conversation that helped children develop understanding about space. In these classrooms, the roles of talk were many and diverse. Talk served to help children develop a mathematical language that fixed and anchored mathematically important elements of space, (e.g., properties of figures). This function of talk played an important role in mathematical generalization: What was

first known primarily through perception, common visual regard, came to be known and shared through talk. Classroom talk was also a vehicle for argument and justification. Much of the classroom talk supported what logicians refer to as suppositional argument: reasoning about "true" propositions purely for the sake of argument. This form of reasoning is sustained by adopting a supposition "for the sake of the argument" and then considering what its consequences would be (Levi, 1996). For example, during the conversation about three rectangles in Ms. C.'s class, one student proposed that form C was a square. She responded, "If C is a square, what must be true?" Suppositional argument requires maintenance of relationships among what is judged true, what is judged false, and what "hangs in suspense." In this instance, children were uncertain about the status of C, although it did look like a square. So they decided to treat C as if it were a square, and then decided that the student's conjecture was false because they found that C's properties were not consistent with those of a square. (One child drew a line congruent with one side and rotated the figure to test for congruence of sides.) Reasoning about the tripartite balance among true, false, and conditional beliefs can be difficult to establish and sustain, even for adults and older children. Yet this form of argument is indispensable to mathematical modeling: What-ifs serve as axioms, and mathematicians explore their consequences.

Classroom conversations also played a key role in helping children articulate a sense of the history of their thinking: Teachers often asked students to narrate how they came to know something. Such narratives helped children develop identities as mathematical thinkers whose mental efforts led to a progressive elaboration of understanding about space. Classroom conversations were the primary means by which mathematical instruction became dialogic, in the sense intended by Bakhtin (1981) and Wertsch (1991). Children came to know as they participated in dialogue, either directly with a peer or with the teacher, or indirectly in relation to some shared supposition (usually established by past practice in the classroom). Classroom talk helped children develop voice (a speaking consciousness) about what they understood first informally and intuitively.

Last, teachers' orchestration of curriculum tasks, tools, notations, and classroom talk was guided by their continually evolving understanding of student thinking. Ms. C.'s design of tasks to promote children's understanding of area and its measure and her continued attempts to help children reflect on their thinking about area and its measure suggest a form of teaching that hinged on her "reading" of student understanding. In each of the classrooms we observed, the evolution of student thinking was matched by a corresponding evolution in teachers' understanding of the pedagogical implications of student thinking.

Our observations of these classroom-based cases of the progressive elaboration of student understanding about space suggest the need for a reexamination of pedagogical policies and practices that ignore the mathematics of space in the primary grades. The opening chapters of this volume indicate that spatial reasoning and visualization are essential to mathematics. In addition to its central role in mathematics, for many children, spatial reasoning provides a more accessible entrée to powerful mathematical ideas like conjecture, proof, and refutation. The lesson seems clear: Space and geometry are best introduced early in schooling and thereafter maintained as a central part of learning and understanding mathematics. But whatever the point of departure, space and number can be mutually constituted only by children who are afforded opportunity to reason about them jointly.

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APPENDIX: EXAMPLES OF ITEMS FOR INDIVIDUAL ASSESSMENT

Area of a Rectangle

Figure 7.A1 displays a paper-and-pencil item designed to assess children's understanding of finding the area of a rectangular polygon.

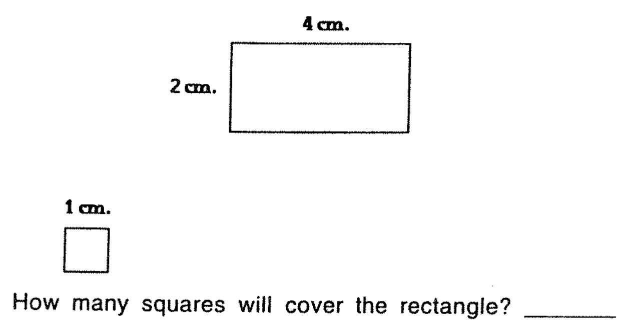

FIG. 7.A1. Paper-and-pencil item for finding area of regular polygon.

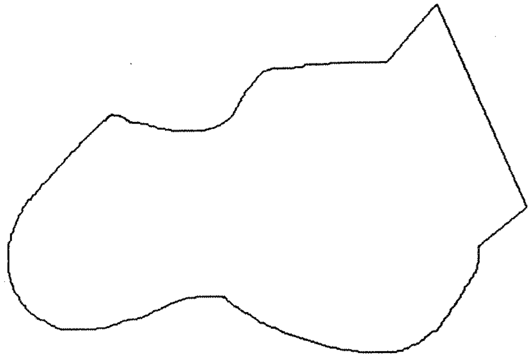

FIG. 7.A2. Interview item on finding area of irregular figures.

Area of Irregular Figure

Students found the area of the irregular figure shown in Fig. 7.A2. Students were offered tools such as a graph paper transparency (divided into squares 2 cm on a side), an overhead marker, a ruler, and a length of string.

Included in the interview protocols were scaffolds to assist students in developing a strategy for solving the problem:

> *Int*: Can you find out how much area this shape has? If the student does not recognize the word "area," ask how much it would take to "cover" exactly this shape.
>
> *Scaffold*: Offer the student the graph transparency and pen and ask, "Would this help you find the answer?"

Figure 7.A3 shows a student's solution (at the end of the school year) to the area problem shown in Fig. 7.A2. Note her strategy of aggregating

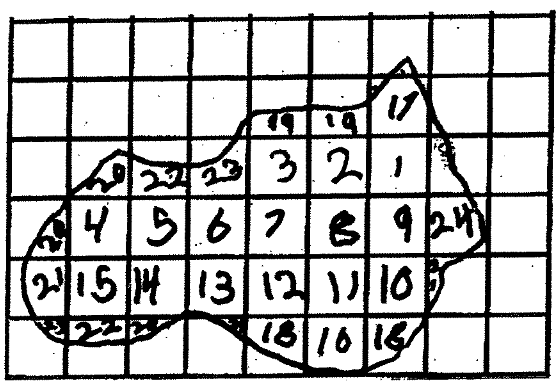

FIG. 7.A3. Example of student solution to finding area of irregular figure.

pieces of the figure that individually did not cover an entire square (e.g., two 19s at the top of the figure, each numbered inside a half-covered square.) By this student's estimate, the area of this figure is 24 squares.

Transformations

Figure 7.A4 displays a paper-and-pencil item designed to assess production of one-step transformations on a simple core square. To probe students' understandings of transformations, we presented strips and core squares (see Fig. 7.A5) and then asked students to compose the transformations that would make the strip.

This item also included several levels of scaffolds—a verbal prompt, an identical core square as a manipulative, and modeling of the movement:

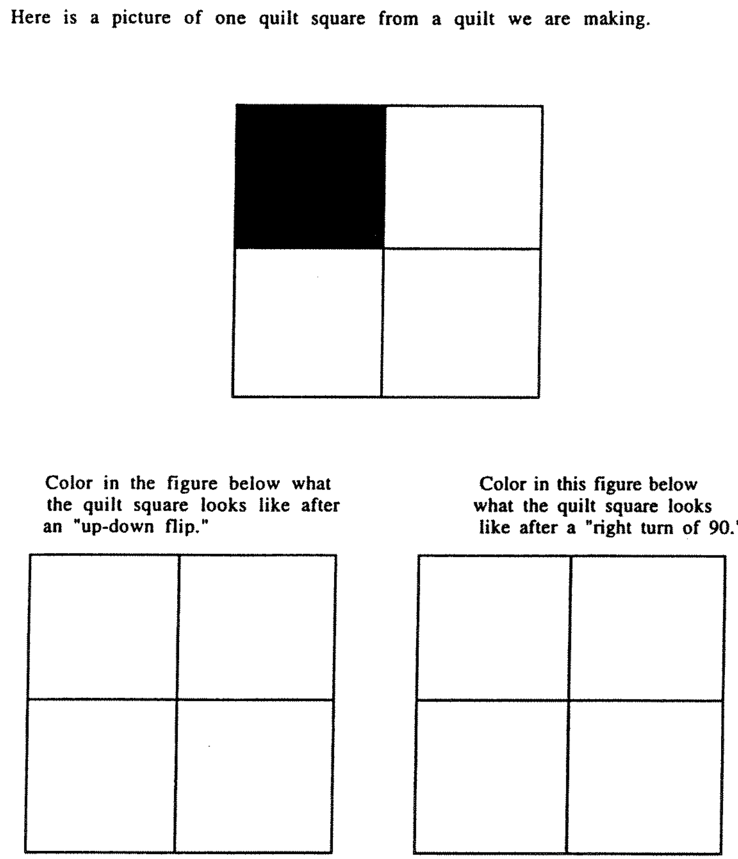

FIG. 7.A4. Pencil-and-paper item on one-step transformations.

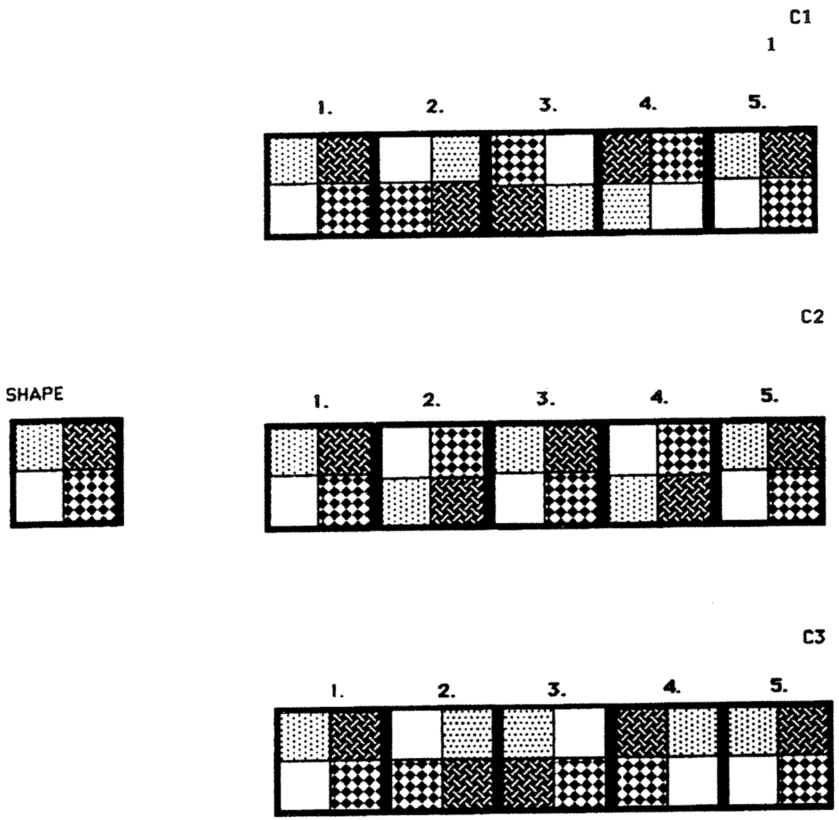

FIG. 7.A5. Interview item that examines students' understandings of transformations, based on recognition rather than on performance.

Int: For each of the motion strips, ask the student, "What did we do to make this design with this core square?" or "Can you find the pattern we used to make this long quilt?"

Scaffolds

Level I "What kinds of motions do you know about?"
"Could it be turns (C1)? flips (C2)? or both (C3)?"

Level II Offer the core square to the student.
"Would this help you to figure it out?"

Level III Model the first movement (only) for the student.
"Do you think you could show me how the rest are made?"

8

Development of Geometric and Measurement Ideas

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The separation of curriculum development, classroom teaching, and mathematics educational research from each other has vitiated each of these efforts. We are working on several related projects, the aim of which is to combine these efforts synergistically. The first[1] is a large-scale curriculum development project that emphasizes meaningful mathematical problems and depth rather than exposure. Our responsibility (and goal) in this project is to develop the geometry and spatial-sense units in this curriculum based on existing research on children's learning of mathematics as well as our own classroom-based research on children's learning within the context of formative evaluations of the curriculum. The second project[2] has the related goal of conducting research on children's learning of geometric and spatial concepts in computer and noncomputer environments.

The curriculum unit, discussed in this chapter, *Turtle Paths* (Clements, Battista, Akers, Woolley, Meredith, & McMillen, 1995), engages third-grade students in a series of combined geometric and arithmetic investigations

[1]*Investigations in Number, Data, and Space: An Elementary Mathematics Curriculum*, a cooperative project among the University of Buffalo, Kent State University, Technical Education Research Center, and Southeastern Massachusetts University (National Science Foundation grant no. ESI-9050210).

[2]*An Investigation of the Development of Elementary Children's Geometric Thinking in Computer and Noncomputer Environments* (National Science Foundation Research grant no. ESI 8954664).